

Review of Major Topics in Physics 402

The basic tools of Quantum Mechanics

Infinite square well $\mathcal{H}^0 = \frac{p^2}{2m} + V(x)$ $V(x) = \begin{cases} 0 & \text{for } 0 < x < a \\ \infty & \text{for } x < 0 \text{ and } x > a \end{cases}$ $E_n^0 = \frac{n^2\pi^2\hbar^2}{2ma^2}$

and $\psi^0(x) = \begin{cases} \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi x}{a}\right) & \text{for } 0 < x < a \\ 0 & \text{for } x < 0 \text{ and } x > a \end{cases}$ with $n = 1, 2, 3, \dots$

Harmonic Oscillator $H^0 = \frac{p_x^2}{2m} + \frac{k}{2}x^2 = \left(a_+a_- + \frac{1}{2}\right)\hbar\omega$, $\psi_n(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} \frac{1}{\sqrt{2^n n!}} H_n(\xi) e^{-\xi^2/2}$, $\xi \equiv \sqrt{\frac{m\omega}{\hbar}}x$, $E_n = (n + \frac{1}{2})\hbar\omega$; $n = 0, 1, 2, \dots$; $x = \sqrt{\frac{\hbar}{2m\omega}}(a_+ + a_-)$; $p = i\sqrt{\frac{\hbar m\omega}{2}}(a_+ - a_-)$;

Raising and lowering operators: $a_+\psi_n = \sqrt{n+1}\psi_{n+1}$; $a_-\psi_n = \sqrt{n}\psi_{n-1}$

Hydrogen Atom $H^0 = -\frac{\hbar^2}{2m}\nabla^2 + \frac{(-e)(+e)}{4\pi\epsilon_0 r}$, Quantum numbers: $n = 1, 2, 3, \dots$, $\ell =$

$0, 1, \dots, n-1, -\ell \leq m \leq \ell$, $\psi_{n,\ell,m}^0(r, \theta, \varphi) = Const_{n,\ell} e^{-\frac{r}{na}} \left(\frac{2r}{na}\right)^\ell L_{n-\ell-1}^{2\ell+1} \left(\frac{2r}{na}\right) Y_\ell^m(\theta, \phi)$;
 $\psi_{100}^0(r, \theta, \varphi) = \frac{2}{\sqrt{4\pi} a^{3/2}} e^{-r/a}$; $\psi_{200}^0(r, \theta, \varphi) = \frac{2}{\sqrt{4\pi} (2a)^{3/2}} (1 - r/(2a)) e^{-r/2a}$;

Energy eigenvalues: $E_n = -13.6 \text{ eV}/n^2$,

Angular Momentum: $L^2|\ell m_\ell\rangle = \ell(\ell+1)\hbar^2|\ell m_\ell\rangle$, $L_z|\ell m_\ell\rangle = m_\ell\hbar|\ell m_\ell\rangle$;

Spin Angular Momentum: $S^2|s m_s\rangle = s(s+1)\hbar^2|s m_s\rangle$, $S_z|s m_s\rangle = m_s\hbar|s m_s\rangle$;

Total Spin Angular Momentum: $\vec{J} = \vec{L} + \vec{S}$, $J^2|j m_j\rangle = j(j+1)\hbar^2|j m_j\rangle$, $J_z|j m_j\rangle = m_j\hbar|j m_j\rangle$.

Code letters: “s” means $\ell = 0$, “p” means $\ell = 1$, “d” means $\ell = 2$, “f” means $\ell = 3$, etc.

Term notation: ${}^{2S+1}L_J$

Spin-1/2 $\vec{S} = \frac{\hbar}{2}(\sigma_x, \sigma_y, \sigma_z)$. $S_x = \frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $S_y = \frac{\hbar}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $S_z = \frac{\hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. page 168

Spin Singlet: $|0\ 0\rangle = \frac{1}{\sqrt{2}}(|\downarrow\rangle|\uparrow\rangle - |\uparrow\rangle|\downarrow\rangle)$;

Spin Triplet: $|1\ -1\rangle = |\uparrow\rangle|\uparrow\rangle$; $|1\ 0\rangle = \frac{1}{\sqrt{2}}(|\downarrow\rangle|\uparrow\rangle + |\uparrow\rangle|\downarrow\rangle)$; $|1\ -1\rangle = |\downarrow\rangle|\downarrow\rangle$

Perturbation theory $H^0\psi_n^0 = E_n^0\psi_n^0$, $H = H^0 + H^1$, $H\psi_n = E_n\psi_n$;

$$\psi_n = \psi_n^0 + \lambda\psi_n^1 + \lambda^2\psi_n^2 + \dots; E_n = E_n^0 + \lambda E_n^1 + \lambda^2 E_n^2 + \dots;$$

First order correction to energy: $E_n^1 = \int \psi_n^{0*} H' \psi_n^0 d^3r$

First order correction to eigenstates: $\psi_n^1 = \sum_{\ell \neq n} \left(\frac{\int \psi_\ell^{0*} H' \psi_n^0 d^3r}{E_n^0 - E_\ell^0} \right) \psi_\ell^0$,

Second order change to energy: $E_n^2 = \sum_{k \neq n} \frac{\left| \int \psi_k^{0*} H' \psi_n^0 d^3r \right|^2}{E_n^0 - E_k^0}$,

Relativistic kinetic energy correction: $H' = -\frac{p^4}{8m^3c^2}$;

$$E_{n,\ell}^1 = -|E_n^0| \frac{\alpha^2}{4n^2} \left[\frac{4n}{\ell + \frac{1}{2}} - 3 \right];$$

Fine structure constant: $\alpha \equiv \frac{e^2}{4\pi\epsilon_0\hbar c} \approx \frac{1}{137.036};$

Degenerate perturbation theory: $\bar{W}\vec{\alpha} = E^1\vec{\alpha}$ $W_{k,j} \equiv \langle \psi_k^0 | H' | \psi_j^0 \rangle$

Spin-orbit interaction: $H_{so} = -\vec{\mu} \bullet \vec{B}; \vec{\mu} = -\frac{e}{m} \vec{S}$ for the electron; $\vec{S} \bullet \vec{L} = \frac{1}{2}(J^2 - L^2 - S^2);$

Spin-orbit energy correction: $E_{n,\ell,s,j}^1 = \frac{|E_n^0|\alpha^2}{n} \frac{j(j+1) - \ell(\ell+1) - 3/4}{2\ell(\ell + \frac{1}{2})(\ell+1)}$;

Fine structure splitting: $\Delta E = E_n^{1\text{Relativity}} + E_n^{1\text{SpinOrbit}} = \frac{|E_n^0|\alpha^2}{n^2} \left[\frac{3}{4} - \frac{n}{j + \frac{1}{2}} \right];$

Transforming between coupled and un-coupled representations – Clebsch-Gordan

coefficients: $|j m_j\rangle = \sum_{m_\ell + m_s = m_j} C_{m_\ell m_s}^{\ell s} |m_j\rangle |\ell m_\ell\rangle |s m_s\rangle;$

Weak field Zeeman effect ($B_{ext} \ll B_{int}$): $\vec{\mu}_{Total} = \vec{\mu}_\ell + \vec{\mu}_s = -\frac{e}{2m} (\vec{L} + 2\vec{S}); \mathcal{H}_Z^1 = -\vec{\mu}_{Total} \cdot \vec{B}_{ext}; E_Z^1 = \frac{e}{2m} \vec{B}_{ext} \cdot (\vec{J} + \vec{S}); E_Z^1 = \mu_B g_J B_{ext} m_J; \mu_B = \frac{e\hbar}{2m};$
Landé g-factor $g_J = 1 + \frac{j(j+1) - \ell(\ell+1) + s(s+1)}{2j(j+1)}$;

Strong field Zeeman effect ($B_{ext} \gg B_{int}$): $E_{Z,strong}^1 = \frac{e}{2m} \vec{B}_{ext} \cdot (\vec{L} + 2\vec{S}) = \mu_B B_{ext} (m_\ell + 2m_s); E_{fs}^1 = \frac{|E_n^0|\alpha^2}{n^3} \left\{ \frac{3}{4n} - \left[\frac{\ell(\ell+1) - m_\ell m_s}{\ell(\ell+1/2)(\ell+1)} \right] \right\};$

Hyperfine interaction (electron magnetic moment interacts with proton magnetic

moment): $H_{HF} = -\vec{\mu}_e \bullet \vec{B}_{dip}; E_{n,0,0}^1 = \frac{\mu_0 g e^2}{3m_e m_p} \frac{\hbar^2}{\pi n^3 a^3} \begin{cases} 1/4 & \text{TRIPLET} \\ -3/4 & \text{SINGLET} \end{cases}$

Many Identical Particles with Overlapping Wavefunctions:

Exchange operator: $\hat{P}\Psi(1,2) = \Psi(2,1), \hat{P}^2 = 1$ Symmetric and anti-symmetric wavefunctions:

$$\Psi_A^0(1,2) = \frac{1}{\sqrt{2}} (\psi_a(1)\psi_b(2) - \psi_a(2)\psi_b(1)) \quad \Psi_S^0(1,2) = \frac{1}{\sqrt{2}} (\psi_a(1)\psi_b(2) + \psi_a(2)\psi_b(1))$$

$$\Psi(1,2) = \pm\Psi(2,1)$$

Bosons and Fermions:

Exchange Correlations built into wavefunctions:

$$\langle (x_1 - x_2)^2 \rangle_{\pm} = \langle x^2 \rangle_a + \langle x^2 \rangle_b - 2\langle x \rangle_a \langle x \rangle_b \mp 2|\langle x \rangle_{ab}|^2 \text{ with } \langle x \rangle_{ab} \equiv \int x \psi_a^*(x) \psi_b(x) dx$$

Feynman-Hellmann theorem: $\frac{\partial E_n}{\partial \lambda} = \langle \psi_n | \frac{\partial \mathcal{H}}{\partial \lambda} | \psi_n \rangle$

Time-Dependent Perturbation Theory: $\mathcal{H}\Psi = i\hbar \frac{\partial \Psi}{\partial t}$ $\Psi(\vec{r}, t) = \psi(\vec{r})e^{-iEt/\hbar}$
 $\dot{c}_a = -\frac{i}{\hbar} \mathcal{H}'_{ab} e^{-i\omega_0 t} c_b, \dot{c}_b = -\frac{i}{\hbar} \mathcal{H}'_{ba} e^{+i\omega_0 t} c_a, \mathcal{H}'_{ab} \equiv \langle \psi_a | \mathcal{H}' | \psi_b \rangle, \omega_0 = (E_b - E_a)/\hbar.$

Two-level system: $c_a(0) = 1, c_b(0) = 0, c_b(t) = -\frac{i}{\hbar} \int_0^t \mathcal{H}'_{ba}(t') e^{i\omega_0 t'} dt'.$

$\dot{a}_{nj} = \frac{-i}{\hbar} e^{i(E_j^0 - E_n^0)t/\hbar} \int \psi_j^*(\vec{x}) \mathcal{H}'(\vec{x}, t) \psi_n(\vec{x}) d^3x$ with $|a_{nj}|^2$ the transition probability from state n to j .

Sinusoidal perturbation: $\mathcal{H}'(\vec{r}, t) = V(\vec{r}) \cos \omega t; P_{a \rightarrow b}(t) = |c_b(t)|^2 \cong \frac{|V_{ab}|^2}{\hbar^2} \frac{\sin^2[(\omega_0 - \omega)t/2]}{(\omega_0 - \omega)^2}$ with $V_{ab} \equiv \langle \psi_a | V(\vec{r}) | \psi_b \rangle$;

Spontaneous emission rate: $A = \frac{\omega_0^3 |\vec{\phi}|^2}{3\pi\varepsilon_0\hbar c^3}$ with $|\vec{\phi}| \equiv q \langle \psi_b | \vec{r} | \psi_a \rangle$;

$$A = \frac{\hbar\omega^3}{\pi^2 c^3} \frac{\pi e^2}{3\varepsilon_0 \hbar^2} \left(|x_{ab}|^2 + |y_{ab}|^2 + |z_{ab}|^2 \right);$$

Electric dipole selection rules: No transitions occur unless $\Delta m = \pm 1$ or 0 and $\Delta \ell = \pm 1$;

$$\begin{cases} \text{if } m' = m, & \text{then } \langle n'\ell'm' | x | n\ell m \rangle = \langle n'\ell'm' | y | n\ell m \rangle = 0 \\ \text{if } m' = m \pm 1, & \text{then } \langle n'\ell'm' | x | n\ell m \rangle = \pm i \langle n'\ell'm' | y | n\ell m \rangle \\ & \text{and } \langle n'\ell'm' | z | n\ell m \rangle = 0 \end{cases}$$

Planck blackbody radiation: $\rho(\omega) = \frac{\hbar\omega^3 / (\pi^2 c^3)}{e^{\hbar\omega/k_B T} - 1}$; Absorption probability due to incoherent

$$\text{light: } R_{a \rightarrow b} = \frac{\pi e^2 \rho(\omega_{ab})}{3\varepsilon_0 \hbar^2} \left(|x_{ab}|^2 + |y_{ab}|^2 + |z_{ab}|^2 \right) \equiv \rho(\omega_{ab}) M_{ab};$$

Fermi's golden rule for transition from a discrete initial state 'i' to a final state in the continuum: $R_{i \rightarrow f} = \frac{2\pi}{\hbar} \left| \frac{\mathcal{H}_{if}}{2} \right|^2 g(E_f)$, where $g(E_f)$ is the density of states at the final energy.

WKB semi-classical approximation: $\psi(x) = \frac{D}{\sqrt{p_{class}(x)}} \exp \left[\pm \frac{i}{\hbar} \int_x^x p_{class}(x') dx' \right];$
 $p_{class} = \sqrt{2m(E - V(x))};$

For infinite square well potentials: $\frac{1}{\hbar} \int_0^a \sqrt{2m(E_n - V(x))} dx = \pi n$, with $n = 1, 2, 3, \dots$;

For finite wells with classical turning points: $\int_{x_1}^{x_2} \sqrt{2m(E_n - V(x))} dx = \pi \hbar \left(n - \frac{1}{2} \right)$, with $n = 1, 2, 3, \dots$ and x_1, x_2 the classical turning points;

For tunneling: $\psi(x) = \frac{D}{\sqrt{|p_{class}(x)|}} \exp\left[\pm \frac{1}{\hbar} \int_0^x |p_{class}(x')| dx'\right]; \quad T \propto e^{-2\gamma}, \quad \text{where}$

$$\gamma = \frac{1}{\hbar} \int_0^a |p_{class}(x')| dx';$$

Fowler-Nordheim tunneling: $T = \exp\left[-\frac{4}{3} \frac{\sqrt{2m}}{\hbar} \frac{\Phi^{3/2}}{e\varepsilon}\right];$

$$\text{Lifetime: } \tau = \frac{2r_1}{v} e^{2\gamma}$$

Variational Method:

Use a guess ground state wavefunction, with embedded parameters $(\lambda_1, \lambda_1, \dots)$ to derive an upper limit on the ground state energy E_{GS} :

$$E_{GS} \leq \langle \Psi_{GS, \text{Guess}} | H | \Psi_{GS, \text{Guess}} \rangle; \quad \frac{\partial \langle \Psi_{GS, \text{Guess}} (\vec{r}, \lambda_1, \lambda_2, \lambda_3, \dots) | H | \Psi_{GS, \text{Guess}} (\vec{r}, \lambda_1, \lambda_2, \lambda_3, \dots) \rangle}{\partial \lambda_i} = 0$$

Quantum Scattering Theory:

N_{scatt} (into $d\Omega$ around θ, φ) = $N_{inc} n_{target} \frac{d\sigma}{d\Omega}(\theta, \varphi) d\Omega$, where $\frac{d\sigma}{d\Omega}(\theta, \varphi)$ is the differential scattering cross section (DSCS).

Total scattering cross section: $\sigma = \iint \frac{d\sigma}{d\Omega}(\theta, \varphi) d\Omega$, $d\Omega = \sin \theta d\theta d\phi$,

Classical scattering theory yields: $\frac{d\sigma}{d\Omega} = \frac{b}{\sin \theta} \left| \frac{db}{d\theta} \right|$, b = impact parameter,

Rutherford scattering: $\frac{d\sigma}{d\Omega} = \left(\frac{qQ/4\pi\varepsilon_0}{4E \sin^2(\theta/2)} \right)^2$.

Quantum scattering for the total wavefunction: $\psi(r, \theta) = A \left\{ e^{ikz} + f(\theta) \frac{e^{ikr}}{r} \right\}$,

$$\frac{d\sigma}{d\Omega} = |f(\theta)|^2,$$

Partial wave analysis: $\psi(r, \theta) = A \left\{ e^{ikz} + k \sum_\ell i^\ell (2\ell + 1) a_\ell h_\ell^{(1)}(kr) P_\ell(\cos \theta) \right\}$,

$$h_\ell^{(1)}(kr) = j_\ell(kr) + i n_\ell(kr) \text{ and } h_\ell^{(2)}(kr) = j_\ell(kr) - i n_\ell(kr),$$

$h_\ell^{(1)}(kr \gg 1) \rightarrow \frac{e^{ikr}}{r}$ an outgoing scattered wave,

$$\sigma = \iint D(\theta) d\Omega = \iint |f(\theta)|^2 d\Omega = 4\pi \sum_{\ell=0}^{\infty} (2\ell + 1) |a_\ell|^2,$$

$$\psi(r, \theta) = A \left\{ \sum_\ell i^\ell (2\ell + 1) [j_\ell(kr) + i k a_\ell h_\ell^{(1)}(kr)] P_\ell(\cos \theta) \right\}.$$

Scattering phase shifts: $a_\ell = \frac{1}{2ik} (e^{i2\delta_\ell} - 1) = \frac{1}{k} e^{i\delta_\ell} \sin \delta_\ell$, $\sigma = \frac{4\pi}{k^2} \sum_{\ell=0}^{\infty} (2\ell + 1) \sin^2(\delta_\ell)$.

Born series: The TISE can be re-written as: $(\nabla^2 + k^2)\psi = Q$, with $E = \frac{\hbar^2 k^2}{2m}$, and $Q \equiv \frac{2m}{\hbar^2} V \psi$,

The Green's function satisfies $(\nabla^2 + k^2)G = \delta^3(\vec{r})$, so that $\psi(\vec{r}) = \int G(\vec{r} - \vec{r}_0) Q(\vec{r}_0) d^3 \vec{r}_0$, with $G(\vec{r}) = -\frac{e^{ikr}}{4\pi r}$.

Lippmann-Schwinger equation for the total wavefunction:

$\psi(\vec{r}) = \psi_0(\vec{r}) - \frac{m}{2\pi\hbar^2} \int \frac{e^{ik|\vec{r}-\vec{r}_0|}}{|\vec{r}-\vec{r}_0|} V(\vec{r}_0) \psi(\vec{r}_0) d^3\vec{r}_0$, where $\psi_0(\vec{r})$ is a free-particle solution.

Scattering amplitude function in the $|\vec{r}| \gg |\vec{r}_0|$ approximation:

$$f(\theta) = -\frac{m}{2\pi\hbar^2 A} \int e^{-i\vec{k}\cdot\vec{r}_0} V(\vec{r}_0) \psi(\vec{r}_0) d^3\vec{r}_0,$$

First Born approximation: $f(\theta) \cong -\frac{m}{2\pi\hbar^2} \int e^{i(\vec{k}'-\vec{k})\cdot\vec{r}_0} V(\vec{r}_0) d^3\vec{r}_0$,

Low energy limit (deBroglie wavelength much larger than the scale of the potential):

$$f_{Low-Energy}(\theta) \approx -\frac{m}{2\pi\hbar^2} \int V(\vec{r}_0) d^3\vec{r}_0,$$

Spherically-symmetric potential:

$$f_{spherically\ symmetric}(\theta) = -\frac{2m}{\hbar^2 \kappa} \int_0^\infty r_0 V(r_0) \sin(\kappa r_0) dr_0, \text{ with } \vec{\kappa} \equiv \vec{k}' - \vec{k}, \text{ and } \kappa = 2k \sin\left(\frac{\theta}{2}\right).$$

Born series expansion $\psi = \psi_0 + \int g V \psi_0 + \iint g V g V \psi_0 + \iiint g V g V g V \psi_0 + \dots$

Free-Electron Fermi Gas: 3D infinite square well ($L_x \times L_y \times L_z = V$) with periodic (Born-von Karmen) boundary conditions: $\psi(x, y, z) \sim \frac{1}{\sqrt{V}} e^{ik_x x} e^{ik_y y} e^{ik_z z}$ with $\vec{k} = 2\pi \left(\frac{n_x}{L_x}, \frac{n_y}{L_y}, \frac{n_z}{L_z} \right)$, $n_x = 0, \pm 1, \pm 2, \pm 3, \dots$, etc.

Momentum $\vec{p} = -i\hbar\vec{\nabla}$ has eigenvalue $\hbar\vec{k} = \hbar(k_x, k_y, k_z)$, momentum space, or k-space

Each state takes up a volume of $\frac{2\pi}{L_x} \times \frac{2\pi}{L_y} \times \frac{2\pi}{L_z} = \frac{(2\pi)^3}{V}$ in k-space.

Fermi wavevector $k_F = (3\pi^2\rho)^{1/3}$, where $\rho \equiv Nq/V$ is the number density of electrons, Fermi energy $E_F = \frac{\hbar^2 k_F^2}{2m} = \frac{\hbar^2}{2m} (3\pi^2\rho)^{2/3}$.

Total energy $U_{Total} = \frac{\hbar^2}{2m} \frac{V}{\pi^2} \frac{1}{5} k_F^5$, degeneracy pressure $P = \frac{2}{3} \frac{U_{Total}}{V} = \frac{\hbar^2}{5m} (3\pi^2)^{2/3} \rho^{5/3}$.

Density of states (DOS) in 3D: The number of single-particle states available between energy E and $E + dE$, $D(E)dE = \frac{V}{2\pi^2} \left(\frac{2m}{\hbar^2} \right)^{3/2} \sqrt{E} dE$.

Cooper pairing problem: Two electrons added outside the filled Fermi sphere, $\left\{ -\frac{\hbar^2}{2m} \nabla_1^2 - \frac{\hbar^2}{2m} \nabla_2^2 + V(\vec{r}_1, \vec{r}_2) \right\} \Psi(1,2) = E \Psi(1,2), \quad \Psi(1,2) = \sum_{k>k_F} g_k \cos(\vec{k} \bullet \vec{r}) |00\rangle$, TISE

becomes $(E - 2\varepsilon_k) g_k = \sum_{k'} g_{k'} V_{k,k'}, \quad \text{with} \quad \varepsilon_k = \hbar^2 k^2 / 2m \quad \text{and}$

$$V_{k,k'} \equiv \int d^3r V(r) \exp[i(\vec{k} - \vec{k}') \bullet \vec{r}].$$

With the attractive interaction $V_{k,k'} = \begin{cases} -V & \text{when } E_F \leq \varepsilon_k \leq E_F + \hbar\omega_c, \text{ where } V > 0 \\ 0 & \text{when } \varepsilon_k > E_F + \hbar\omega_c \end{cases}$, one

finds a bound state: $E \cong 2E_F - 2\hbar\omega_c e^{-2/(N(E_F)V)}$, where $N(E_F)$ is the DOS at E_F .

Electrons in a Periodic Potential: The Kronig-Penney Model: Periodic potential of finite square wells of width b , depth V_0 and periodicity a such that $V(x+a) = V(x)$.

Bloch's theorem: $\psi(x) = e^{iqx}u(x)$, where q (crystal momentum) is a real number and $u(x)$ has the same periodicity as the potential.

Translation operator: $\hat{T}(a)\psi(x) = \psi(x - a)$. with $\hat{T}(a) = e^{-ia\hat{p}/\hbar}$. For any operator \hat{Q} : $\hat{T}(a)^\dagger \hat{Q}(\hat{x}, \hat{p}) \hat{T}(a) = \hat{Q}(\hat{x} + a, \hat{p})$.

For the Kronig-Penney Hamiltonian: $\hat{\mathcal{H}}(\hat{x}, \hat{p})\psi(x) = E\psi(x)$ and $\hat{T}(a)\psi(x) = \lambda\psi(x) \equiv e^{-iqa}\psi(x)$.

In the deep and narrow well approximation: $\frac{-\beta^2 ba}{2} \frac{\sin(\alpha a)}{\alpha a} + \cos(\alpha a) = \cos(qa)$ with $\alpha = \sqrt{\frac{2mE}{\hbar^2}}$, $\beta = \sqrt{\frac{2m(E+V_0)}{\hbar^2}}$, and q the crystal momentum. The dispersion relation $E = E(q)$ shows bands and band gaps.

Superfluid ${}^4\text{He}$ and Bose-Einstein condensation: Identical Bosons in a box solved with standing wave boundary conditions: $\psi(x, y, z) = A \sin\left(\frac{n_x \pi x}{a}\right) \sin\left(\frac{n_y \pi y}{a}\right) \sin\left(\frac{n_z \pi z}{a}\right)$ with $n_x = 1, 2, 3, \dots$, etc.

Energy level occupation in equilibrium at temperature T with chemical potential μ (result from quantum statistical mechanics): $n_s = \frac{g_s}{e^{(E_s - \mu)/k_B T} - 1}$, with $s = (n_x, n_y, n_z)$.

Enforcing the fixed number of particles (N) constraint yields

$$N = n_1(T) + 2\pi V \left(\frac{2m}{\hbar^2}\right)^{3/2} (k_B T)^{3/2} \Gamma\left(\frac{3}{2}\right) f(\mu/k_B T) \text{ with } f(\mu/k_B T) \equiv \sum_{p=1}^{\infty} \frac{e^{p\mu/k_B T}}{p^{3/2}}$$

Critical temperature for Bose-Einstein condensation:
$$T_c = \left(\frac{N/V}{2\pi \left(\frac{2mk_B}{\hbar^2}\right)^{3/2} \Gamma\left(\frac{3}{2}\right) 2.612} \right)^{2/3}$$
.

Macroscopic quantum wavefunction $\psi(\vec{r}) = \sqrt{\rho_s(\vec{r})} e^{i\theta(\vec{r})}$ and quantized circulation

$$\kappa = \oint \vec{v}_s \bullet d\vec{l} = \frac{nh}{m}$$
, where $n = 0, \pm 1, \pm 2, \dots$